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A discussion of PIBAL Report No. 167

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BUCKLING OF A CIRCULAR CYLINDER UNDER UNIFORM PRESSURE

A Discussion of PIBAL Report Number 167 and of
Remarks on that Report by Dr. E. H. Kennard

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TABLE OF CONTENTS

SUMMARY

A report, PIBAL number 167, was written by Salerno and Levine on the buckling of cylinders and titled "Buckling of Circular Cylindrical Shells with Evenly Spaced, Equal Strength Circular Ring Frames. Part I."

In discussing that report Dr. Kennard pointed out certain apparent inconsistencies in theory.

~~In this present report~~ the material presented in PIBAL ~~number~~ No. 167 is discussed together with the remarks made by Dr. Kennard. It appears that some of the theory presented in PIBAL ~~number~~ No. 167 is open to question but, nevertheless, the conclusions obtained therein hold without modification.

This report is divided into three main parts. In part 1 is presented some material on the theory of instability, and some of the quantities contributing to the total potential energy for the current shell problem are derived in detail. In part 2 is discussed the essential differences between the work of Salerno and Levine and of Kennard. It is shown that although some of the work of the former authors is open to question and to criticism the terms representing the difference between their work and that of Kennard make no essential difference to the concluding eigenvalve formula. In part 3 the effects of the terms in question in both treatments are traced and considered in detail. As one conclusion it appears that certain terms proposed by Kennard, although admissible, lead to a result different from that obtained by Bryan for the limiting case of an infinite cylinder.

LIST OF SYMBOLS

R = radius of middle surface of shell

h = thickness of shell

L = distance between ring frames

L_o = L/R

V = volume occupied by shell

E = modulus of elasticity

ν = Poisson's ratio

x, s = axial and circumferential coordinates

φ = angular coordinate circumferentially

ξ = x/R

u, v, w = displacements of shell

κ = curvature

χ = change in curvature

σ = stress

e = strain

N_{xx}, N_{ss}, N_{xs} = stress resultants per unit length

M_{xx}, M_{ss}, M_{xs} = moment resultants per unit length

p = radial pressure

P = axial pressure

U_a = strain energy arising from stretching of middle surface

U_b = strain energy arising from membrane shear stress

U_c = strain energy arising from bending moments and torques induced in change of state

U_d = strain energy arising from work done by external force system

U_e = strain energy stored in the reinforcing rings

W_2 = work done by radial pressure

V_1 = potential due to axial pressure

m = number of half waves circumferentially

n = number of half waves axially

λ = $n\Pi R/L$

R_A, R_{AC}, R_C = ring parameters

Φ_1 = pR/K

Φ_2 = Ph/K

k = $h^2/12R^2$

K = $Eh/(1-\nu^2)$

$C_1 \cdots\cdots C_7$ = coefficients arising in expansion of determinants

PART 1. GENERAL THEORY

1. Instability Conditions.

We consider a closed, right circular cylinder in an initial state of zero stress. The cylinder is assumed to be very long in comparison with its diameter, and to contain many similar bays formed by a set of equi-spaced internal circular reinforcing rings.

Let this shell be now loaded by an external hydrostatic force system. We desire to compute the magnitude of the pressure at which the cylinder will buckle between adjacent ring frames.

(i) Let the hydrostatic load system be applied to the unstressed shell in such a manner that, although on the point of buckling, the cylinder has not actually buckled. Let the associated hydrostatic pressure be p.

We will call this stressed state the state A. It is an equilibrium state.

In the usual theoretical approach to a shell problem of the present type it is generally assumed that the cylinder is compressed uniformly by the hydrostatic pressure before the reinforcing rings are positioned. Thus in state A cylinder generators are straight.

(ii) Now, without alteration in the magnitude of the hydrostatic load system, we enforce a small change in the shell shape so that it takes on an adjacent bent shape. We will call the associated stress-state state B. In state B the generators are no longer

straight.

Such an adjacent deformation could be induced, for instance, by the action of a small additional force system acting normal to the circumferential wall of the shell.

If, on the removal of this small additional force system, the shell remains deformed in the adjacent configuration, then we say that the hydrostatic pressure p is the buckling pressure for this shell structure.

For the buckling pressure p, state B as just described (with the small additional force system removed) is also an equilibrium state.

(iii) Let the total potential energy (i.e. strain energy minus work done by external loads which are assumed as constant in going from initial to final deflected shape) associated with state A be Π_A.

Let the change in potential energy in going from state A to state B be Π_B.

Let the total potential energy in state B be Π, measured from zero stress content.

Then, since state B is an equilibrium state, the Potential Energy Theorem of Elasticity states that

$$\Pi = \Pi_A + \Pi_B$$

is stationary. Hence, for small variations in displacements

$$\delta\Pi = \delta(\Pi_A + \Pi_B) = 0.$$

But Π_A is also an equilibrium state, so that

$$\delta\Pi_A = 0.$$

Thus, for p to be the buckling pressure, we require

$$\delta\Pi_B = 0. \tag{1}$$

(iv) We can obtain another result from a slightly different viewpoint. We know state A to be an equilibrium state and, for the p associated with state A to be an instability pressure, we ask for the existence of a neighboring state which is also an equilibrium state under the action of the same load system. For this to be so the change in potential energy in going from state A to state B must be zero. Hence

$$\Pi_B = 0. \tag{2}$$

We shall use condition (1) as the criterion by means of which we will evaluate, approximately, the buckling pressure p. We could equally well, of course, use the criterion

$$\delta\Pi = 0.$$

2. Potential Energy Π_B.

It is convenient to consider the strain energy of the stressed shell as consisting of various contributions: those due to middle surface stretching and shearing, and those due to bending and twisting. To do this it is necessary to be able to write down quantities representing changes in curvature, etc., in going

from state A to state B.

It would, perhaps, be better to consider the strain energy in a purely formal manner, and write it as

$$E_e = \frac{1}{2}\int_V \sigma_{ij}e_{ij}dV \ , \quad (i,j, = 1,2,3)$$

where the usual summation convention is implied, and

$$\sigma_{ij} = \sigma_{ij}\ (e_{ij})$$

via the stress-strain relations. To do this, however, would require a detailed knowledge of all the strain components throughout the shell wall expressed in terms of displacement components. This approach has not been used in PIBAL number 167, and so will not be used here.

The potential energy, then, will be taken to consist of five contributions:

(a) That arising from stretching of the middle surface in going from the "plane" state A to the adjacent "bent" state B. Stipulating state B to be an adjacent state implies that it is very close to state A even though the deformation pattern is different. As such it is assumed that the changes in magnitude of the normal membrane stresses are small in comparison with their magnitudes in state A. In fact, frequently it is assumed that the normal membrane stresses remain constant in going from state A to state B. We do not do so here, however.

(b) That arising from membrane shear stresses. In state A - a state of uniform compression - the membrane shear stresses are zero; in state B they are finite though small. Hence, in computing the membrane shear stress contribution to Π_B, cognizance

should be taken of this growth. The strain energy contribution will be

$$U_s = \frac{1}{2}\int_V (\text{shear stress})_B(\text{engineering shear strain})_B \, dV.$$

Again, as with the variation in the normal membrane stresses in going from state A to state B, frequently this quantity is regarded as small in comparison with Π_B and is neglected. Nevertheless, we will retain this term.

(c) That arising from bending moments and torques induced in the change of state.

(d) That arising from the work done by the external force system while the structure deforms from the "plane" state to the adjacent "bent" state. During this deformation the external hydrostatic pressure remains constant in magnitude (and even if it did not it would be assumed that it did), so that the work done is equal to p times the change in volume occupied by the shell.

(e) Strain energy in the reinforcing rings.

3. Computation of Potential Energy Contributions.

Since we only are concerned with state B it is convenient to take stressed state A as datum for purposes of measurement of the additional displacements involved in going to the "adjacent" state. Accordingly, we assume that in state A the cylinder is of radius R, wall thickness h, and is of length L center to center of ring frames.

Let u,v,w be the displacement components in state B measured from zero in state A. These displacements are taken as positive according to the scheme

u in the direction of x increasing,

v in the direction of s increasing,

w in the direction of r increasing,

where the coordinate system x,s,r is as shown in Fig. 1. As can be seen, w is thus positive when measured in the direction of the inward normal to the shell curved surface.

Let the energy contributions associated with (a), (b), (c), (d), (e) of § 2 be U_a, U_b, U_c, U_d, U_e, respectively. Then, with barred quantities denoting the stress and moment resultants (per unit length of arc on the middle surface of shell wall) existing in state A, and unbarred quantities denoting the changes in those resultants in going from state A to state B, we have:

$$\text{(a):} \quad U_a = \int_A (\bar{N}_{xx}e_{xx} + \bar{N}_{ss}e_{ss})dA + \frac{1}{2}\int_A (N_{xx}e_{xx} + N_{ss}e_{ss})dA \, , \tag{3}$$

where

$$\left\{ \begin{array}{l} \bar{N}_{xx} = \text{constant} \, , \\ \bar{N}_{ss} = \text{constant} \, , \end{array} \right\} \tag{4}$$

$$\left\{ \begin{array}{l} N_{xx} = N_{xx}(e_{xx}) \, , \\ N_{ss} = N_{ss}(e_{ss}) \, , \end{array} \right\} \tag{5}$$

and the region of integration A is the surface area of the circumferential "middle surface" of the shell wall.

$$\text{(b):} \qquad U_b = \frac{1}{2}\int_A N_{xs}e_{xs}dA \, , \tag{6}$$

where

$$N_{xs} = N_{xs}(e_{xs}) \, , \tag{7}$$

and e_{xs} is the engineering (i.e. not the tensor component) shear strain at the middle surface.

(c):
$$U_c = \frac{1}{2}\int_A (M_{xx}\varkappa_{xx} + M_{ss}\varkappa_{ss} + M_{xs}\varkappa_{xs})dA \qquad (8)$$

where
$$M_{xx} = M_{xx}(\varkappa_{xx},\varkappa_{ss}) \text{ , etc.} \qquad (9)$$

(d):
$$U_d = p.\Delta V \qquad (10)$$

where ΔV is the decrease in the volume occupied by the shell in going from state A to state B.

(e): We will not consider U_e in detail in this paper.

4. Detailed Computation.

From static considerations we know that

$$\left.\begin{aligned} \overline{N}_{xx} &= -Rp/2 \text{ , i.e. compression,} \\ \overline{N}_{ss} &= -Rp \text{ ,} \end{aligned}\right\} \qquad (11)$$

where as usual p is the applied hydrostatic pressure.

Also, from the strain-displacement relations for a circular cylinder, we have that

$$\left.\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \text{ ,} \\ e_{ss} &= \frac{\partial v}{\partial s} - \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial s}\right)^2 \text{,} \\ e_{xs} &= \left(\frac{\partial u}{\partial s} + \frac{\partial v}{\partial x}\right) + \frac{\partial w}{\partial s}\frac{\partial w}{\partial x} \text{ .} \end{aligned}\right\} \qquad (12)$$

In addition, we have the stress-strain relations

$$\sigma_{xx} = [E/(1-\nu^2)](e_{xx} + \nu e_{ss}) \quad ,$$

$$\sigma_{ss} = [E/(1-\nu^2)](e_{ss} + \nu e_{xx}) \quad ,$$

$$\sigma_{xs} = [E/2(1+\nu)]e_{xs} \quad .$$

so that the stress-resultants are given by

$$\left.\begin{aligned} N_{xx} &= [E/(1-\nu^2)](e_{xx} + \nu e_{ss}) \quad , \\ N_{ss} &= [E/(1-\nu^2)](e_{ss} + \nu e_{xx}) \quad , \\ N_{xs} &= [Eh/2(1+\nu)]e_{xs} \quad . \end{aligned}\right\} \tag{13}$$

Substituting (12) into (13) then gives the stress-resultants in terms of displacements.

In the second integral in expression (3) we normally disregard terms of degree higher than the second so as to obtain a linear problem when finding approximate solutions by means of the Rayleigh-Ritz technique. On so doing, we obtain, therefore,

$$\begin{aligned} U_a = &- (Rp/2) \int_A \{u_x + 2v_s - 2(w/R) + (1/2)w_x{}^2 + w_s{}^2\}dA \\ &+[Eh/2(1-\nu^2)] \int_A \{u_x^2+v_s^2+(w/R)^2-2v_s(w/R)+2\nu(u_x v_s-u_x(w/R))\}dA \end{aligned} \tag{14}$$

where now for convenience we use the notation

$$u_x = \frac{\partial u}{\partial x} \text{ , etc.}$$

For U_b we have:

$$U_b = (1/2) \int_A N_{xs} e_{xs} dA ,$$

$$= (1/2)[Eh/2(1+\nu)] \int_A (u_s + v_x + w_x w_s)^2 dA , \qquad (15)$$

$$= (1/2)[Eh/2(1+\nu)] \int_A (u_s + v_x)^2 dA \qquad (15a)$$

on neglecting terms of degree higher than two.

To conpute U_c we need expressions for the quantities $\varkappa_{xx}, \cdots,$ in terms of displacement components u,v,w, and quantities $M_{xx}, \cdots,$ in terms of curvatures or displacements via stress-strain relations. Since these terms have not been mentioned in the discussion by Kennard we assume the results of PIBAL number 167 without comment or elaboration (PIBAL number 167, page 4), and so have

$$U_c = (D/2) \int_A \{(\chi_x + \chi_\varphi)^2 - 2(1-\nu)(\chi_x \chi_\varphi - f^2)\} dA ,$$

where

$$\chi_x = w_{xx} ,$$

$$\chi_\varphi = w_{ss} + w/R^2 ,$$

$$f = w_{xs} + (v_x/2R) - (u_s/2R) .$$

Hence

$$U_c = [Eh^3/24(1-\nu^2)] \int_A \{(w_{xx} + w_{ss} + w/R^2)^2$$

$$- 2(1-\nu)[w_{xx} w_{ss} + w \cdot w_{xx}/R^2 - (w_{xs} + v_x/2R - u_s/2R)^2]\} dA. \qquad (16)$$

To compute U_d the contributions from the end forces and from the radial forces will be considered separately. We note that the end thrust produces (among other things) a stress resultant $\bar{N}_{xx}$ of magnitude $-Rp/2$. If, now, we consider an element of the curved surface of the shell, of length dx and (circumferential) width ds, and defined longitudinally by the two points AB as shown in Fig. 2, we have:

displacement in x direction of point A = u_A ,

displacement in x direction of point B = $u_A + du_A$.

Hence, for the element in question, the work done by $\bar{N}_{xx}$ is given by

$$\Delta U_{de} = (Rp/2)[u_A - (u_A + du_A)]ds ,$$

and so

$$U_{de} = -(Rp/2)\int_A u_x dA . \tag{17}$$

In passing, and although not strictly relevent here, we note that if the cylinder were short and closed at the ends the work done by the end thrust would be

$$U_{de} = P \int_{end} [u]_{x=}^{x=0} dA .$$

For the second contribution to U_d we note that if any normal force $\bar{p}$ at a given point on the curved surface of the shell moved with the point during the displacement of the point on taking up state B, then (and as is usually done in Elasticity) the work done by such a force at that point would be, simply,

$$\bar{U}_{dr} = \bar{p}\, w .$$

Following Kennard*, however, we consider this matter further.

5. Work done by Lateral Hydrostatic Pressure.

In more detail, we consider a point C located on a lateral surface generator in state A, and displaced to a position D in state B, Fig. 3. Let D' be the projection of D on the generator passing through C. Then, if the normal applied force moved with C, its normal displacement would be w_c. If, however, the force originally at C remained in that same position and did not move with C, its normal displacement would be $[w]_c$. This we can take to be the situation as far as the lateral surface hydrostatic pressure is concerned. From Fig. 3, on allowing for the circumferential displacement v_c of point C (not shown on the figure), we see that to within a first derivative correction $[w]_c$ is given by

$$[w]_c = w_c - u_c\,(w_x)_c - v_c\,(w_s)_c\ .$$

Using this result we can compute the change in volume of a cylinder of original radius R and length L. The original volume is $\int_0^L \Pi R^2 dx$. The final volume is

$$V = \int_0^L \left\{(1/2)\int_0^{2\Pi}[R - (w - uw_x - vw_s)]^2 d\varphi\right\}dx\ ,$$

or

$$V = \int_0^L \left\{(1/2R)\int_0^{2\Pi R}[R - (w - uw_x - vw_s)]^2 ds\right\}dx\ .$$

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* Kennard's discussion on PIBAL number 167 is included for convenience in this report as Appendix IV.

Thus the change in volume is given by

$$\bar{V} = \int_A \{w - (w^2/2R) - uw_x - vw_s\} dA$$

where terms of degree higher than the second have been neglected.

Hence we see that the work done by the radial forces is

$$U_{dr} = p \int_A \{w - (w^2/2R) - uw_x - vw_s\} dA \,, \qquad (18)$$

where, as usual, A is the area of lateral or curved (middle) surface of the shell in state A.

To all the foregoing we should, for completeness, add energy terms due to strain in the end reinforcing rings frames. This we will not do here, however; and, when such terms are required we will take them direct from PIBAL number 167.

With the above material as framework we are now in a position to discuss material contained in PIBAL number 167 together with the remarks on that material by Dr. Kennard.

PART 2. DISCUSSION

6. Quantity $(U_a - U_d)$ derived here and by Kennard.

In his work Dr. Kennard obtained some energy quantities which differ from those given in PIBAL number 167. Essentially, the differences occur in the energy contributions U_a and U_d. We consider, then, the quantity $U_a - U_d$.

From (14), (17), and (18) we have

$$
\begin{aligned}
U_a - U_d = & -(Rp/2) \int_A \{u_x + 2v_s - (2w/R) + (w_x{}^2/2) + w_s{}^2\} dA \\
& + [Eh/2(1-\nu^2)] \int_A \{u_x^2 + v_s^2 + (w^2/R^2) - (2v_s w/R) + 2\nu(u_x v_s - u_x w/R)\} dA \\
& + (Rp/2) \int_A u_x dA - (Rp/2) \int_A \{(2w/R) - (w^2/R^2) - (2uw_x/R) - (2vw_s/R)\} dA.
\end{aligned}
$$

Further, we note that v is periodic in the circumferential direction, with a consequence that $\int_A v_s dA$ has magnitude zero. Thus we have

$$
U_a - U_d = Rp \int_A \{(w^2/2R^2) + (uw_x/R) + (vw_s/R) - (w_x^2/4) - (w_s^2/2)\} dA + (\Delta U_1)_b \quad .
$$

where

$$
(\Delta U_1)_b = [Eh/2(1-\nu^2)] \int_A \{u_x^2 + v_s^2 + (w^2/R^2) - (2v_s w/R) + 2\nu u_x v_s - (2\nu u_x w/R)\} dA; \tag{19}
$$

This is the result obtained from the foregoing considerations only.

On making obvious changes in notation, expression (19) is in complete agreement with the corresponding expression (ε) given by Kennard.

7. Quantity (U_a-U_d) obtained from PIBAL number 167.

After a little manipulation the quantity (U_a-U_d) obtained from the work in PIBAL number 167 is given by

$$U_a-U_d = Rp \int_A \Big\{ -(w/R)+(w^2/2R^2)+(w\ w_{ss}/2)-(w\ u_x/2R)$$

$$- (w_x{}^2/4)-(v_x{}^2/4)-(u_x{}^2/4)\Big\} dA + (\Delta U_1)_b \ . \qquad (20)$$

Note that as written here this is not the expression (ε') quoted by Kennard in his work as the PIBAL result in question.

It should be remarked here that in (19) no account has been taken of the membrane shear stress contribution U_b. On the other hand, as obtained from PIBAL number 167 as will be shown shortly, (20) comes from (but does not include) quantities which take into consideration such effects.

From a comparison of (19) and (20) there are obvious differences, and it would appear that the latter expression is incorrect. Nevertheless, although a portion of the theoretical treatment presented in PIBAL number 167 seems in error, it will be shown later that the results and conclusions given in that report remain essentially the same as those obtained from (19). Before doing this it is of interest to discuss the deviation of (20) further.

The PIBAL number 167 equivalent of (20) is made up from equations* $(\overline{3})$, $(\overline{20})$ and $(\overline{25})$ of that report. For completeness

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* The numbers of the PIBAL report equations are here shown as barred numbers for clarity. In that report the numbers are the same but they are not barred.

they are repeated here. We have

$$\text{(i)}\quad U_e = [Eh/2(1-\nu^2)]\int_0^{2\pi}\int_0^{L_o}\left\{u_\xi^2 + (v_\varphi - w)^2 + 2\nu u_\xi(v_\varphi - w) + (1-\nu)(v_\xi + u_\varphi)^2/2\right\} d\xi\, d\varphi \qquad (\overline{3})$$

where

U_e is "the energy of stretching",

$\xi = x/a$ or, in our notation, $= x/R$,

$L_o = L/R$;

$$\text{(ii)}\quad V_1 = -(Ph/2)\int_0^{2\pi}\int_0^{L_o}(u_\xi^2 + v_\xi^2 + w_\xi^2)\, d\xi\, d\varphi\ , \qquad (\overline{20})$$

where

V_1 is the "potential caused by axial pressure", and, on interpretation of quantities given on Pibal pages 17 and 19,

$$P = pR/2h,$$

and on page 9 of the Pibal report P is defined as the "axial pressure";

$$\text{(iii)}\qquad W_2 = (1/2)\int_0^{2\pi}\int_0^{L} pw\left\{2 + u_x - (w + w_{\varphi\varphi})/R\right\} R\, dx\, d\varphi\ , \qquad (\overline{25})$$

where

W_2 is the "work done by the radial pressure."

We note that $(\overline{3})$ as just presented and as used in Pibal number 167 represents the total strain energy due to <u>all</u> membrane stresses, i.e. shear as well as normal, and we see that V_1 as defined above is the negative of the work done by the pressure acting on the ends of the cylinder.

In virture of the definition of U_e just given, it follows that U_e represents our expression $(U_a + U_b)$. In consequence of this, and the fact that U_b is not present in (19), it is convenient to write U_e as

$$U_e = U_{e_1} + U_{e_2} \quad ,$$

where

$$U_{e_1} = [Eh/2(1-\nu^2)] \int_0^{2\pi} \int_0^{L_o} \{ u_\xi^2 + (v_\varphi - w)^2 + 2\nu\, u_\xi (v_\varphi - w) \} d\xi \; d\varphi \quad ,$$

$$U_{e_2} = [Eh/2(1-\nu^2)] \int_0^{2\pi} \int_0^{L} \{ (1-\nu)(v_\xi + u_\varphi)^2/2 \} \; d\xi \; d\varphi \quad .$$

To transform these to our notation we use the substitutions

$$\xi = x/R, \qquad s = R\varphi \quad ,$$

from which

$$d\xi = dx/R, \qquad d\varphi = ds/R \quad ,$$

and, on noting that $L_o = L/R$, it follows that

$$U_{e_2} = [Eh/4(1+\nu)] \int_0^{2\pi R} \int_0^{L} (Rv_x + Ru_s)^2 \; dx \; ds/R^2 \quad ,$$

$$= [Eh/4(1+\nu)] \int_0^{2\pi R} \int_0^{L} (v_x + u_s)^2 \; dx \; ds \quad ,$$

$$= U_b \quad .$$

Consequently, in obtaining (20), i.e. the equivalent of $(U_a - U_d)$, we must use U_{e_1} instead of U_e .

Using the same transformations for ξ and φ we obtain for U_{e_1} the expression

$$U_{e_1} = [Eh/2(1-\nu^2)] \int_0^{2\pi R} \int_0^L \left\{R^2 u_x^2 + (Rv_s - w)^2 + 2\nu R u_x (Rv_s - w)\right\} dx\, ds/R^2 ,$$

$$= [Eh/2(1-\nu^2)] \int_0^{2\pi R} \int_0^L \left\{u_x^2 + v_s^2 - (2wv_s/R) + (w^2/R^2) + 2\nu u_x v_s - 2\nu w u_x/R\right\} dx\, ds . \tag{21}$$

In similar manner we can compute $V_1 - W_2$, noting that we require this difference since V_1 is the negative of the work done by the "end force". We have

$$V_1 - W_2 = -(Ph/2) \int_0^{2\pi} \int_0^{L_0} (u_\xi^2 + v_\xi^2 + w_\xi^2)\, d\xi\, d\varphi$$

$$- (1/2) \int_0^{2\pi} \int_0^L pw \left\{2 + u_x - (w + w_{\varphi\varphi})/R\right\} R\, dx\, d\varphi ,$$

and, on noting that

$$Ph/2 = (pR/2h)(h/2) = pR/4 ,$$

this becomes

$$V_1 - W_2 = -(pR/4) \int_0^{2\pi R} \int_0^L \left\{(R u_x)^2 + (R v_x)^2 + (R w_x)^2\right\} dx\, ds/R^2$$

$$- (1/2) \int_0^{2\pi R} \int_0^L pw \left\{2 + u_x - (w + R^2 w_{ss})/R\right\} dx\, ds,$$

$$= -pR \int_0^{2\pi R} \int_0^L \left[\left\{(u_x^2/4) + (v_x^2/4) + (w_x^2/4)\right\} + w\left\{2 + u_x - (w + R^2 w_{ss})/R\right\}/2R\right] d$$

$$= pR \int_0^{2\pi R} \int_0^L \left\{-(u_x^2/4) - (v_x^2/4) - (w_x^2/4) - (w/R) - (wu_x/2R) + (w^2/2R^2) + (ww_{ss}/2)\right\} d$$

(22)

Thus, on comparison we see finally that (21) is the same as $(\Delta U_1)_b$ of (20) and (22) is the same as the quantity under the integral sign in (20), so that the quantity from Pibal number 167 to be compared with (U_a-U_d), namely $U_{e_1} + V_1 - W_2$, is given by (20) as was stated.

8. Motivation of derivation in Pibal number 167.

As the authors of Pibal report number 167 are now not available, it cannot be stated with any certainty what were the bases on which the expressions contributing to (20) were derived.

It is conjectured that for U_e of Pibal number 167, that is the energy contributions from all membrane stresses, only the second integral term in (3) was used instead of the full two-term expression of (3); in addition, of course, the quantity U_b as given by (6) was also used. The second term of (3) rerpesents energy due only to the change in magnitude of the normal membrane stresses in going from state A to state B, and the U_e of Pibal number 167 apparently neglects the existence of normal membrane stresses already present in state A.

In the expressions for the work done by the end forces, the Pibal number 167 expressions can be derived by assuming that the complete non-linear strain expression, namely

$$e_{xx} = u_x + (u_x^2 + v_x^2 + w_x^2)/2 \quad ,$$

holds, and then by imposing the condition that the buckling process is essentially inextensional, i.e. $e_{xx} = 0$, etc. It is usually considered

that the non-linear terms u_x^2, v_x^2 are of higher order than w_x^2 , and in the Pibal report all terms arising from these were subsequently eliminated from the final results. That this was the manner of derivation in the Pibal work is again pure conjecture.

The Pibal expression for the work done by the radial forces, i.e. $(\overline{25})$, is open to criticism in that it seems that to compute the local change in radius of the original circular cylinder the radius of curvature of the deformed cylinder (state B) was used. If this was so, it is incorrect.

9. Effect of errors in original derivation.

Having obtained (20), and discussed the possible manner of the original derivation of that expression, we now consider the difference, D say, between (20) and (19). Writing

$$D = (19) - (20)$$

we have

$$D = Rp\int_A \Big[\Big\{ (w\, u_x/2R) + (v_x^2/4) + (u_x^2/4) - (w\, w_{ss}/2) + (w/R) \Big\} + \Big\{ (u\, w_x/R) + (v\, w_s/R) - (w_s^2/2) \Big\} \Big] dA \quad ,$$

where the quantity in the first set of curly braces consists of the terms left from Pibal number 167, and that in the second set of curly braces is from Kennard's work when terms common to (19) and (20) are cancelled.

As will be discussed in detail later, in the first quantity certain individual terms are subsequently neglected in Pibal number 167 on the

basis of a smallness argument. The terms so neglected are

$$w\, u_x/2R \quad , \quad v_x^2/4 \quad , \quad u_x^2/4 \quad .$$

In a similar manner, that is for comparable accuracy (again to be discussed in detail later) two terms in the second quantity can also be neglected. These are

$$u\, w_x/R \quad , \quad v\, w_s/R \quad .$$

Thus, essentially, the difference between (19) and (20) is given by

$$D = -(Rp/2) \int_A w\, w_{ss} + w_s^2 - 2w/R \quad dA \; . \tag{23}$$

From Pibal number 167 the chosen displacement coefficient functions were

$$u = A \cos m\varphi \;\; \cos \lambda\xi \;\; = A \cos\,(ms/R) \cos\,(\lambda x/R) \quad ,$$

$$v = B \sin m\varphi \;\; \sin \lambda\xi \;\; = B \sin\,(ms/R) \sin\,(\lambda x/R) \quad , \tag{$\overline{27}$}$$

$$w = C \cos m\varphi \;\; \sin \lambda\xi \;\; = C \cos\,(ms/R) \sin\,(\lambda x/R) \quad ,$$

where m is the number of curcumferential half-waves,

$$\lambda = n\pi R/L \quad ,$$

and n is the number of half-waves in the axial directio.

Substituting ($\overline{27}$) in (23) we obtain the result mentioned earlier,

$$D = -(Rp/2) \int_A \left\{ w\, w_{ss} + w_s^2 - 2w/R \right\} dA = 0 \quad . \tag{23a}$$

This is discussed in more detail in Appendix III.

Thus we conclude that, although in no way a justification for the theoretical treatment given in Pibal number 167, and depending on a smallness argument still to be presented, the simplified formula ($\overline{48}$) given in the Pibal report stands. The formula is

$$\Phi_1 = \frac{\lambda^2(1-\nu^2)/(m^2+\lambda^2) \;+\; R_c \;+\; k(m^2+\lambda^2)}{(m^2-1+\lambda^2/2)} , \qquad (\overline{48})$$

where the various quantities and symbols concernedwill be mentioned in detail later.

It can further be stated that the numerical work relating to Pibal number 167 was based on equation ($\overline{48}$).

The energy terms of Pibal number 167 were used in other Pibal reports, namely Pibal numbers 169, 177, 182 and 189. These reports extend the previous work by considering other shell constraints at the frames, by discussing various aspects of shell behavior and (in 189) by considering the overall buckling of a reinforced shell between bulkheads. In all reports deflection functions are used which satisfy equation (23a). Smallness arguments similar to those presented here are used in these reports in the derivation of the simplified formulas which were used for numerical calculations. On this basis it seems that the working formulas and numerical work of the Pibal reports using the energy expressions of Pibal number 167 are valid.

PART 3. DETECTIVE WORK

10. Essential steps in obtaining $(\overline{48})$ of Pibal number 167.

Having shown that, on neglecting certain terms, the main result obtained in Pibal number 167, namely $(\overline{48})$, still holds, we now discuss the justification of the approximations resorted to. To do so it is necessary to outline the work presented in the Pibal report.

In the main body of the present report it has not been emphasized that the cylinder being considered is reinforced with ring frames and, in fact, expressions for the strain energy in the rings have not yet been given. These we will not discuss here. We present, instead, the complete expression for the total potential energy as used in Pibal number 167. Note that this includes the supposedly incorrect contribution (20). Consequently, after discussing the methodology and manipulation used in the Pibal report, we will then draw corresponding conclusions obtained on using (19) instead of (20).

From Pibal number 167 we have for the total potential energy the expression

$$U_T = U_e + U_b + U_r + V_1 + V_2 \qquad (\overline{34})$$

where the individual contributions $U_e, \ldots, V_2$ are given by $(\overline{3})$, $(\overline{4})$, $(\overline{13})$, $(\overline{20})$ and $(\overline{26})$ respectively. Here

U_T = total "potential", i.e. total potential energy,

U_e = "strain energy of stretching", i.e. membrane stresses strain energy,

U_b = "strain energy of bending",

U_r = "energy stored in a ring",

V_1 = "potential caused by the axial pressure",

V_2 = "potential energy due to the radial pressure".

In Pibal number 167 the work then follows the usual motivation of the Rayleigh-Ritz process for computing approximations for eigenvalues. The chosen displacement functions $(\overline{27})$ are inserted in $(\overline{34})$ and the indicated integration carried out.

Conditions necessary for rendering U_T stationary, namely

$$\partial U_T/\partial A = \partial U_T/\partial B = \partial U_T/\partial C = 0 \quad , \qquad (\overline{35})$$

are then imposed. These lead to a set of three simultaneous, homogeneous, algebraic equations linear in A, B, C. For a non-trivial solution to exist the familiar condition that the determinant of the coefficients of A, B, C must be zero is then used, which leads to the eigenvalue determinantal equation $(\overline{39})$, namely

$$\begin{vmatrix} \begin{Bmatrix} \lambda^2 + m^2(1-\nu)(1+k)(1/2) \\ + R_A - \lambda^2\Phi_2 \end{Bmatrix} & \{-\lambda m[(1/2)(1+\nu)-(k/2)(1-\nu)]\} & \begin{Bmatrix} \lambda\nu-(1-\nu)\lambda k m^2 \\ + R_{AC} + \Phi_1\lambda/2 \end{Bmatrix} \\ \{-\lambda m[(1/2)(1+\nu)-(k/2)(1-\nu)]\} & \{m^2+(1-\nu)(1+k)(\lambda^2/2)-\lambda^2\Phi_2\} & \{-m[1+(1-\nu)\lambda^2 k]\} \\ \begin{Bmatrix} \lambda\nu-(1-\nu)\lambda k m^2 \\ + R_{AC} + \Phi_1\lambda/2 \end{Bmatrix} & \{-m[1+(1-\nu)\lambda^2 k]\} & \begin{Bmatrix} 1+k[(m^2+\lambda^2)+1-2m^2-2\nu\lambda^2] \\ + R_c - \lambda^2\Phi_2 - (m^2-1)\Phi_1 \end{Bmatrix} \end{vmatrix} \qquad (\overline{39})$$

It is then shown in Pibal number 167 that, for a cylinder for which the distance between adjacent ring frames becomes large ($\to\infty$), the solution

of $(\overline{39})$ yields the classic Bryan solution for an unstiffened cylinder:

$$p = (m^2 - 1)\left\{Eh^3/12R^3(1-\nu^2)\right\} . \qquad (\overline{44})$$

In this treatment various quantities were introduced, in particular we have

$$K = Eh/(1-\nu^2) \quad , \qquad k = h^2/12R^2 \quad ; \qquad (\overline{29})$$

$$\Phi_1 = p\,R/K \quad (= pR(1-\nu^2)/Eh) \quad ,$$

$$\Phi_2 = P\,h/K \quad (= pR(1-\nu^2)/2Eh)$$

for the hydrostatic loading case discussed here, so that

$$\Phi_1/2 = \Phi_2 \quad . \qquad (\overline{43})$$

As mentioned in Pibal number 167, "the complete expansion of the determinant $(\overline{39})$ leads to a cubic equation in Φ_1 and Φ_2." Consequently as a first approximation (see Appendix 1), in the expansion powers and products of Φ_1, Φ_2, and k greater than the first can be neglected. This leads to the approximate expansion

$$C_1 + C_2 k = C_3\Phi_1 + C_4\Phi_2 \quad , \qquad (\overline{42})$$

or, using $(\overline{43})$,

$$C_1 + C_2 k = (C_3 + C_4/2)\Phi_1 \quad . \qquad (\overline{44})$$

Expressions for $C_1, \ldots, C_4$ are given by $(\overline{44}a), \ldots, (\overline{44}d)$ of Pibal number 167. On inspection of these it is apparent that for m large (e.g. $m > 8$) each expression contains one term dominant in comparison with the remaining terms of that expression. This dominant term is independent of the ring

parameters. In the expression for C_1 there also occurs a second dominant term of similar magnitude but involving the ring parameter R_c.

If only the dominant terms are retained and used in ($\overline{44}$) we obtain, finally, the approximate result ($\overline{48}$). Further, if in this equation the term containing the ring parameter is deleted, the simplified von Mises result follows, ($\overline{45}$). Actually there is a small discrepancy here, for in lieu of the denominator term $\left\{m^2 - 1 + (\lambda^2/2)\right\}$ of ($\overline{45}$), the von Mise's result neglects the -1 contribution in comparison with m^2.

11. Justification or neglect of certain small Pibal terms.

The foregoing discussion retraces the main steps taken in Pibal number 167 in obtaining ($\overline{48}$). In the process many small quantities were neglected. Among them are terms related to the terms mentioned earlier, i.e.

$$w\, u_x/2R \ , \qquad v_x^2/4 \ , \qquad u_x^2/4 \quad .$$

We now discuss these three terms in more detail.

If ($\overline{39}$) is written as

$$|a_{ij}| = 0 \qquad , \qquad i,j = 1,2.3 \quad ,$$

it is easy to trace the following connections;

(i) In a_{11} the quantity $\lambda^2\Phi_2$ comes directly from the term u_ξ^2 of ($\overline{20}$), that is from u_x^2.

(ii) In a_{22} the quantity $\lambda^2\Phi_2$ comes directly from the term v_ξ^2 of ($\overline{20}$), that is from v_x^2.

(iii) In a_{13} and a_{31} the quantity $\lambda\Phi_1/2$ comes directly from the term $w\,u_\xi/2$ of ($\overline{26}$), that is from $w\,u_x/2R$.

Further, it is simple to show that the R term drops out from the denominators of the terms u_x^2 , v_x^2 .

In the expansion of $(\overline{39})$ these three quantities lead to terms in Φ_1 and Φ_2 of degree higher than the first (and so can be neglected) and to contributions in the non-dominant terms of C_3 and C_4 . They make no contribution to the dominant terms retained in deriving $(\overline{48})$. That this is so is shown in detail in Appendix I.

12. Eigenvalue Determinant for the Kennard Energy terms.

Having discussed the technique and reasoning used in Pibal number 167, we now apply the same thoughts and degree of approximation to the energy quantities deduced by Kennard.

For the total potential energy U_T we have

$$U_T = U_a + U_b + U_c - U_d + U_e \quad ,$$

where

U_a is the energy due to the normal membrane stresses, (14),

U_b is the energy due to the shear membrane stresses, (15a),

U_c is the energy due to the moments, (16),

U_d is the work done by the external force system, (17) + (18),

U_e is the energy due to ring deformation, $(\overline{13})$.

Thus, in this U_T we use expressions from Pibal number 167 for U_c and U_e , after transforming them to x,s coordinates.

On following the procedure prescribed in §10, in lieu of (39) we obtain the determinantal equation

$$\begin{vmatrix} \{\lambda^2 + m^2(1-\nu)(1+k)/2 + R_A\} & -\lambda m\{(1+\nu)-k(1-\nu)\}/2 & \{\lambda\nu - k\lambda m^2(1-\nu) + R_{AC} + \lambda\Phi_1\} \\ -\lambda m\{(1+\nu)-k(1+\nu)\}/2 & \{m^2+(1-\nu)(1+k)\lambda^2/2\} & -m[1+\Phi_1+k\lambda^2(1-\nu)] \\ \{-\lambda\nu - k\lambda m^2(1-\nu) + R_{AC} + \lambda\Phi_1\} & -m[1+\Phi_1+k\lambda^2(1-\nu)] & \{1+k[(m^2+\lambda^2)^2+1-2m^2-2\nu\lambda^2] + R_C - (m^2-1+\lambda^2/2)\Phi_1\} \end{vmatrix} \qquad (24)$$

The complete expansion of (24) leads to a quadratic equation in Φ_1 and Φ_2, where those quantities are the same as previously defined. Again, as a first approximation, powers and products of Φ_1 and k can be neglected in the expansion and, so doing, we obtain

$$C_5 + C_6 k = C_7 \Phi_1 \quad , \qquad (25)$$

Here C_5 and C_6 are new coefficients corresponding to C_1 and C_2 of Pibal number 167, and the new coefficient C_7 takes the place of the previous $(C_3 + C_4/2)$. Expressions for C_5, C_6, C_7 are given in Appendix 2.

As before, on inspection of these expressions it is apparent that for m large (> 8, say) each one contains one dominant term which is independent of the ring parameters, and in C_5 there is also a second term of similar magnitude involving the ring parameter R_C. Further, it can be seen that these dominant terms are the same as those in the expressions for $C_1, \ldots, C_4$.

Again, on retaining only the dominant terms in C_5, C_6, C_7, (25) leads directly to the previous result, (48).

13. Justification for neglect of certain small Kennard terms.

In the immediately preceding derivation of ($\overline{48}$) many quantities have been neglected. Among those so treated are quantities arising from the terms

$$u\, w_x/R \quad , \quad v\, w_s/R \quad .$$

As before, we now discuss these two terms in detail.

If we write (24) as

$$|b_{ij}| = 0 \quad , \quad i,j = 1,2,3,$$

it is a simple matter to trace the following connections:

(i) In b_{13} and b_{31} the quantity $\lambda\Phi_1$ comes directly from the term $u\, w_x/R$;

(ii) In b_{23} and b_{32} the quantity $m\Phi_1$ comes directly from the term $v\, w_s/R$.

In the expansion of (24) these quantities lead to second degree terms in Φ_1 and Φ_2 (and so can be neglected), and to contributions to the non-dominant terms in C_7 . They make no contribution to the dominant terms retained in deriving ($\overline{48}$). This is shown in Appendix II.

14. The Kennard Determinant and the Bryan Solution.

We consider, now, the effect on the Kennard determinantal equation (24) of allowing the distance between the ring frames to become large. On letting $L \to \infty$ or, what is the same thing, $\lambda \to 0$ in (24) we obtain

$$m^2 \left\{ 1 - (1 + \Phi_1)^2 + \Phi_1 (1-m^2) + k(m^2 - 1)^2 \right\} = 0 \; . \tag{26}$$

If we expand this, neglect Φ_1^2 in comparison with Φ_1, and substitute for Φ_1 in terms of p, we obtain

$$p = \left\{m^2 - 3 + 4/(m^2+1)\right\}\left\{ Eh^3/12R^3(1-\nu^2)\right\} ,$$

which is not the Bryan solution $(\overline{41})$. If, however, we write simply

$$(1+\Phi_1)^2 \approx 1$$

in (26), we do obtain the Bryan solution. Obviously in so doing we are not approximating consistently.

The term producing this inconsistency can easily be traced back to earlier considerations. In (26) the term $(1+\Phi_1)^2$ comes from the terms b_{23}, b_{32} in (24) on letting $\lambda \rightarrow 0$. And in b_{23}, b_{32} the quantity Φ_1 in the term $(1+\Phi_1)^2$ comes directly from the term $v\, w_s/R$ which was introduced originally in the expression for the work done by the lateral thrust. The other term of similar character introduced in that same expression, namely $u\, w_x/R$ does not survive here since its end product in (24), namely $\lambda\Phi_1$, vanishes with λ.

15. Further Discussion of Work Done by External Forces.

The computation of the work done by external forces warrants further discussion. We have just seen that use of the expression

$$U_{dr} = p \int_A \{ w - w^2/2R - u\, w_x - v\, w_s \} dA \quad , \qquad (18) \text{ bis}$$

for the work done by the forces normal to the lateral curved surface of the shell leads, apparently, to an inconsitency in comparison with a result

obtained by Bryan. The inconsistency arises not from an error but rather from an order of magnitude disagreement.

The terms $u\,w_x$, $v\,w_s$ refer the final position, after deformation to state B, of a point on the lateral surface back to its original position, and so allow for a more correct integration than is usually the case (in so far that a slope correction has been added) with respect to x over the range $0 \leq x \leq L$ which defined the cylinder in state A. Nevertheless, even when using non-linear strain components, this correction is rarely used in Elasticity problems. Usually, even though deformation has taken place, no allowance is made for the change in area of an element of surface and integration is carried out over regions defined by original unstrained boundary positions.

The Potential Energy Theorem, used in Elasticity and here, states that for an equilibrium state the potential energy* U

$$U = E_G - \int_{S_B} T_i\, u_i\, dS \quad , \qquad i = 1, 2, 3, \tag{27}$$

is stationary for arbitrary variation in displacements. Here

$$E_G = \frac{1}{2} \int_V \sigma_{ij}\, e_{ij}\; dV \quad , \quad i,j = 1,2,3$$

with

$$\sigma_{ij} = \sigma_{ij}\,(e_{ij})$$

via the stress-strain law, and T_i , u_i , $i = 1,2,3$, are the components of

* For the present we refer all quantities to an orthogonal Cartesian coordinate system; see Sokolnikoff "Theory of Elasticity".

surface traction and of the corresponding surface displacement. The portion of the surface designated as S_B is that portion for which

$$\int_s u_i \quad \delta T_i \, dS = 0 \quad .$$

In particular if surface tractions are specified over the entire surface of the body in question, then $\delta T_i \equiv 0$, and $S_B \equiv$ entire surface.

In these various expressions the usual summation convention is understood, and in all applications the limits of integration are the original dimensions of the unstressed elastic body.

If we formally apply theorem (27) to the present problem the term corresponding to the "work done by the hydrostatic pressure" on the lateral surface of the shell is

$$U'_{dr} = \int_A p \, w \, dA \tag{28}$$

where, as usual throughout this paper, $dA = ds\,dx$, and A refers to the curved surface area of the shell in state A.

An expression such as the last represents a computation correct only to the first order and, truly, the Kennard correction terms added to the normal displacement w yields a quantity U_{dr} which is more accurate than is (28).

As an alternative to the Kennard computation one could, of course, compute the work done by the lateral throust by using the formula

$$U''_{dr} = \int_{A''} p \, w \, dA$$

where now, on writing $dA = dx\,ds$, the range of definition of x would be $0 \leq x \leq L - \Delta L$, where ΔL is the shortening of the cylinder on going from

state A to state B and is given by

$$\Delta L = u_L - u_o \quad .$$

However, if the quantity U''_{dr} were used in this connection, then similar limits of integration should also be used in all other computations throughout this work. This, of course, has not been done.

It seems, then, that on retaining the terms

$$u\, w_x \, , \quad v\, w_s$$

in the expression for U_{dr}, computation throughout the treatment is not consistent in accuracy.

Again, in retaining the limits $0 \leq s \leq 2\pi R$, the term $w^2/2R$ in (18) refers to the change in the length ds of the element on going from state A to state B. Thus, for accuracy consistent with that in all other surface area computations this term, likewise, should not be used.

16. Consequence of ds correction term.

Since in all the foregoing work the term $w^2/2R$ has been retained it is of interest to trace its consequences, and so discover the effect of ignoring it.

On substituting the deflection expressions $(\overline{27})$ in the total potential energy expression $(\overline{34})$ and performing the differentiation with respect to the arbitrary parameters A, B and C, the term $w^2/2R$ leads exclusively to the quantity (−1) in the coefficient of Φ_1 in the a_{33} term of the determinantal

equation $(\overline{39})$. The quantity persists, and appears in the denominator of the final simplified expression for the critical pressure of a reinforced cylinder, $(\overline{48})$.

For the case of a cylinder with simple end supports, which amounts to setting the ring parameter term of $(\overline{48})$ equal to zero, the simplified equation of von Mises is obtained with the -1 retained in the denominator. von Mises then drops the -1 in comparison with terms m^2 and λ^2 in order to compensate partially for his previous approximations.

Thus, if the term $w^2/2R$ is not used, the effect would be that the -1 in the denominator of (48) would never appear, so giving (on suitable modification) the final simplified von Mises result directly. Nevertheless, as is well known the approximate von Mises result does not reduce to the Bryan solution.

It is of interest to note that the value of the critical pressure as obtained from the Donnell equation for circular cylindrical shells agrees with the simplified von Mises result; the -1 term does not appear in the denominator.

APPENDIX I

Discussion of Approximations made in Pibal number 167.

1. The complete expansion of the general determinant (39) leads to a cubic equation in Φ_1, Φ_2 and k. As an approximation the equation was linearized and took the form

$$C_1 + C_2 k = C_3 \Phi_1 + C_4 \Phi_2 \quad .$$

We now discuss the validity of neglecting the terms containing powers of Φ_1, Φ_2 and k of order higher than unity.

By definition

$$k = (h/R)^2/12 \quad ,$$

and

$$\Phi_1 = 2\,\Phi_2 = Rp/K = (p/E)(R/h)(1-\nu^2) \quad .$$

By definition of a thin shell, the thickness h must be small compared with the radius R, i.e. $(h/R) \ll 1$. Hence

$$k^2 = (h/R)^4/144 \ll k \ll 1.$$

Similarly

$$k^3 \ll k^2 \quad .$$

The buckling pressure p must always be small compared with the modulus of elasticity E of the material of which the shell is constructed. Also, for the range of shell dimensions considered in this work, the

ratio $(p/D)(R/h) \ll 1$. In the numerical example quoted in Pibal number 167 the figures were

$$h/R = 0.006068\ ,$$

$$P_{crit} = 970 \text{ p.s.i}\ ,$$

together with

$$k = 3{\cdot}068 \times 10^{-6}\ ,$$

$$\Phi_1 = 4{\cdot}847 \times 10^{-3}\ , \quad (\text{from } \overline{48}),$$

$$\Phi_2 = 2{\cdot}424 \times 10^{-3}\ ,$$

which justifies the linearization for this case. Similar results follow for the ranges of shell and frame geometry which formed the bases of charts presented in subsequent reports.

2. In the body of the report it was stated that the terms $w\,u_x/2R$, $v_x^2/4$ and $u_x^2/4$ in (20) when integrated led to terms of order higher than those given by the remainder of the terms under the integral. Consequently they were ignored.

Consider the term $w\,u_x/2R$ in (20). It can be shown that this quantity leads only to the term $\Phi_1\lambda/2$ in the elements a_{13} and a_{31} of the general determinant $(\overline{39})$. Upon expansion of the determinant this term contributes to C_3, that is the coefficient of Φ_1. The minors of the terms a_{13} and a_{31} show that the contributions to C_3 by this term are of order $m^2\lambda$, $m^4\lambda k$, $m^2\lambda^3 k$, with other quantities of lower order in m and λ. On the other hand, examination of the whole expression for C_3, $(\overline{44}c)$ of Pibal number 167, shows that terms exist of order $m^2\lambda^4$, $m^4\lambda^2$, m^6, with other terms

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in m and λ of lower order. It will be shown in the following that the latter terms, $m^2\lambda^4$, $m^4\lambda^2$, m^6, are alone dominant in C_3.

Similar considerations will also show that the terms $v_x^2/4$, $u_x^2/4$ lead to terms in C_3 of lower order in m and λ than the so-called dominant terms.

To show that the higher powers of m and λ dominate in the expression for C_3, and indeed in the expressions for C_1, C_2, and C_4 as well, we recall the definitions of m and λ. Thus, m is the number of circumferential half-waves, while $\lambda = (n\pi R)/L$, where R/L is the radius of the shell divided by the distance between reinforcing rings, and n is the number of longitudinal half-waves.

For shells of the type considered here, i.e. with closely spaced rings, the R/L ratio is of the order of 2 to 6. In the expression for C_1 ($\overline{44}$a) of Pibal number 167) the first two terms on the right hand side contain λ^4 terms, while the remaining terms contain λ raised to a lower power. Consequently, all but the first two terms can be neglected.

Similar considerations, and the fact that $k \ll 1$, indicate that for C_2, C_3, and C_4, all terms other than the first can be neglected.

For shells with large R/L ratio it would be reasonable to expect the number m of circumferential lobes to be large. Thus terms containing the higher powers of m would dominate in the expressions for the coefficients $C_1, \ldots, C_4$. It so happens that the terms already neglected in virtue of the foregoing discussion are also the terms of lower powers of m than those retained.

We conclude, therefore, that the final expressions for $C_1, \ldots, C_4$ which can safely be used for computing the critical pressure are as given by ($\overline{47}$) of Pibal number 167. Note that these expressions are only valid for large values of m.

APPENDIX II.

Discussion of Approximations made to the Kennard expressions.

1. In the body of the report it was stated that, corresponding to neglect of the Pibal terms $w\,u_x/R$, $v_x^2/4$, $u_x^2/4$, the Kennard terms $u\,w_x/R$ and $v\,w_s/R$ can likewise be neglected.

Consider the term $u\,w_x/R$ in the integral of (19). As was stated earlier, this term leads to quantities in the coefficient C_7 which are negligible compared with the results of other terms in that integral.

It can easily be seen that the quantity $u\,w_x/R$ leads to the term $\Phi_1\lambda$ in the elements b_{13} and b_{31} of the general determinant (24). As in Appendix I, we again consider the minors of these two elements. They lead to terms of order $m^2\lambda$, $m^4\lambda k$, $m^2\lambda^3 k$ together with lower order terms. Examination of the total expression for C_7 (see below) shows the dominant terms to be of order $m^2\lambda^4$, $m^4\lambda^2$ and m^6 . We conclude that omission of the term $u\,w_x/R$ negligibly affects the buckling pressure.

2. It was shown in the body of the report that the quantity Φ_1 in the elements b_{23} and b_{32} of the determinant (24) must be neglected in order to arrive at Bryan's result for the buckling of a cylinder of infinite length. This quantity can be shown to result from the term $v\,w_s/R$ in the integral of (19). However, even if the term was retained it would only lead to non-dominant terms inthe expression for C_7 .

3. The expressions for C_5 , C_6 and C_7 , resulting from the expansion of the determinant (24), are:

$$C_5 = [(1-\nu)/2]\left\{\lambda^4(1-\nu^2) + R_c(m^2+\lambda^2)^2 + R_cR_A(\lambda^2 + \frac{2m^2}{1-\nu})\right.$$
$$\left. + \lambda^2R_A + 2\lambda(m^2-\nu\lambda^2)R_{AC} - R_{AC}^2[\lambda^2 + 2m^2/(1-\nu)]\right\} ,$$

$$C_6 = [(1-\nu)/2]\Big\{(m^2+\lambda^2)^4 + 2(m^2+\lambda^2)^2(1-m^2-\nu\lambda^2) - 4(m^2+\lambda^2)(m^2\lambda^2)(1-\nu)$$

$$-(m^2+\nu\lambda^2)^2+\lambda^2(1-4m^2)R_A + R_A(\lambda^2+\frac{2m^2}{1-\nu})[(m^2+\lambda^2)^2 + 1-2m^2 - 2\nu\lambda^2]$$

$$+ 2\lambda(2m^4 + 2m^2\lambda^2 - m^2 - \nu\lambda^2) R_{AC} - \lambda^2 R_{AC}^2$$

$$+ R_C[(m^2+\lambda^2)^2 + \lambda^2 R_A]\Big\} \quad ,$$

$$C_7 = [(1-\nu)/2]\Big\{(m^2-1+\lambda^2/2)(m^2+\lambda^2)^2 - 2\lambda^2(m^2+\nu\lambda^2)$$

$$+ k(m^2-1+\lambda^2/2)(m^2+\lambda^2)^2 - 2m^2\lambda^2(2m^2+2\lambda^2-1)$$

$$+ [2/(1-\nu)][m^2+\frac{\lambda^2}{2}(1-\nu)(1+k)][R_A(m^2-1+\frac{\lambda^2}{2}) + 2R_{AC}]\Big\} \quad .$$

The various simplifications of these coefficients are the same as those outlined in § 2 of Appendix I.

APPENDIX III.

Remaining Differences between Pibal number 167 and the present report.

It was stated in the body of the report that essentially the differences between Pibal number 167 and the present report reduce to showing that (19) and (20) are virtually the same numerically, that is

$$(19) - (20) = 0 \ .$$

After accounting for several of the terms in this equation the remaining difference D was shown to be

$$D = -(1/2)Rp \int_A (w_s^2 + w\,w_{ss} - 2w/R)\,dA \ ,$$

$$= -(1/2)Rp \int_0^{2\pi R} \int_0^L (w_s^2 + w\,w_{ss} - 2w/R)\,dx\,ds. \qquad (a)$$

The deflection function w assumed in Pibal number 167 was

$$w = C \cos \frac{m}{R} \sin \frac{\lambda x}{L} \ . \qquad (b)$$

Consider the term $\frac{w}{R}$ on the right hand side of (a). Using (b) it can be rewritten as

$$\int_0^{2\pi R} \int_0^L (w/R)\,dx\,ds = \frac{c}{R} \int_0^{2\pi R} \int_0^L \cos \frac{ms}{R} \sin(\lambda x/L)\,dx\,ds = 0 \ ,$$

This term would also be zero if w were any deflection function of the form

$$\left.\begin{aligned} w &= X(x)\cos(ms/R) \\ \text{or} \quad w &= X(x)\sin(ms/R) \end{aligned}\right\} \qquad (c)$$

where $X(x)$ is any suitably continuous function of x.

next, consider the remaining two terms of (a). For the assumed function (b) we have

$$\int_A w_s^2 \, dA = - \int_A w \, w_{ss} \, dA$$

so that the two terms cancel each other. This result is also true for any other deflection functions w of the form (c).

We conclude, therefore, that for any w of the form (c), the final results of this report coincide numerically with the final results given in Pibal number 167.

APPENDIX IV

Buckling of a Cylinder under Uniform Pressure

Note by E. H. Kennard

I. Fundamentals

Coordinates on median surface:

x, $s = a\varphi$, a = radius,

φ = angle about axis.

By assumptions of Salerno and Levine (PIBAL 167), in equilibrium state just before buckling the stresses are, P denoting "axial pressure,"

$$\sigma_{xo} = -P, \qquad \sigma_{so} = -2ap/2h = -ap/h.$$

In comparison with these, the radial stress is neglected.
Hence equilibrium strains just before buckling are

$$\varepsilon_{xo} = (1/E)(-P + \nu\, ap/h), \qquad \varepsilon_{so} = (1/E)(\nu P - ap/h).$$

Now the elastic energy will contain certain terms involving only the equilibrium strains, others involving only the added buckling strains, and finally cross products between these two partial strains. Buckling is controlled by terms in the energy and in the external work (or in the "potential") that are quadratic in the buckling displacements or their derivatives. Hence, in general, terms in the buckling strains that are quadratic in this sense must be retained, since in the cross products they make a quadratic contribution to the elastic energy. However, quadratic strains containing u_x(i.e. $\partial u/\partial x$), v_s($\partial v/\partial x$) or w need not be retained, for squares of

these quantities occur in energy terms arising from the buckling strains alone, and in comparison with these terms any cross products quadratic in u_x, v_s, or w will contain also an equilibrium strain as a very small factor, so that such cross products are negligible for practical purposes. It is otherwise with w_x and w_s.

An adequate calculation of the strains after buckling is, therefore,

$$\varepsilon_x = [(1+\varepsilon_{xo}+u_x)^2\,dx^2 + w_x^2\,dx^2]^{1/2}/dx - 1 \;,$$

$$\varepsilon_o = [(1+\varepsilon_{so}+v_s - w/a)^2\,ds^2 + w_s^2\,ds^2]^{1/2}/ds - 1 \;,$$

whence

$$\varepsilon_x = \varepsilon_{xo} + u_x + w_x^2/2, \qquad \varepsilon_s = \varepsilon_{so} + v_s - w/a + w_s^2/2 \;.$$

Let U_1 = extensional elastic energy, $E_s = E/(1-\nu^2)$.

Then

$$U_1 = (hE_s/2)\iint(\varepsilon_x^2 + \varepsilon_s^2 + 2\nu\,\varepsilon_x\,\varepsilon_s)\,dx\,ds \;.$$

The value of U_1 just before buckling is

$$(hE_s/2)\iint(\varepsilon_{xo}^2 + \varepsilon_{so}^2 + 2\nu\,\varepsilon_{xo}\,\varepsilon_{so})\,dx\,ds.$$

The increase in U_1 by buckling or ΔU_1 is, therefore, through terms of 2d order in u and w,

$$\begin{aligned}\Delta U_1 = (hE_s/2)\iint[\,&\varepsilon_{xo}(2u_x + 2\nu v_s - 2\nu w/a + w_x^2 + \nu w_s^2)\\ &+ \varepsilon_{so}(2\nu u_x + 2v_s - 2w/a + \nu w_x^2 + w_s^2)\\ &+ u_x^2 + v_s^2 + w^2/a^2 - 2v_s w/a + 2\nu u_x v_s - 2\nu u_x w/a]\,dx\,ds.\end{aligned}$$

II. Special case: $w = 0$ at ends.

Let it be assumed that the ends of the cylinder remain plane. Then

$$\iint u_x \, dxds = \int (u_L - u_o) ds, \qquad \begin{aligned} u_L &= u \text{ at } x = L , \\ u_o &= u \text{ at } x = 0 , \end{aligned}$$

Also,

$$\iint v_s \, dxds = \int dx \int_0^{2\pi a} v_s ds = 0 .$$

Hence, substituting for ε_{xo} and ε_{xs} ,

$$\left. \begin{aligned} \Delta U_1 = (hE_s/2)\iint(u_x^2 + v_s^2 + w^2/a^2 + 2\nu u_x v_s - 2\nu u_x w/a \\ - 2 v_s w/a)dxds + ap\iint (w/a - w_s^2/2)dxds \\ - \frac{1}{2} hP\iint w_x^2 \, dxds - hP \int_0^{2\pi a} (u_L - u_o)ds. \end{aligned} \right\} \quad (\alpha)$$

[For comparison with PIBAL 167, change notation thus:

$$x \rightarrow a\,\xi \;, \qquad s \rightarrow a\,\varphi \;.]$$

[Term in E_s equals 1st 3 terms of PIBAL 167-(3); last term in (3) is shear energy, not here considered. Terms in p and P in Δu_1 represent cross-product terms which are omitted in PIBAL 167.]

Work by end forces. If $w = 0$ at the ends (but not otherwise, since 2d-order terms in the displacement must be included)

$$W_1 = - hP \int_0^{2\pi a} (u_L - u_o) \, ds \qquad (\beta)$$

(provided P is uniform around the shell).

Here P is assumed constant during the displacement.

[PIBAL 167 has, from (20),

$$W_1 = -V_1 = (hP/2)\int\int (u_x^2 + v_s^2 + w_s^2)dxds. \qquad (\beta')$$

The deduction of (20) is not understood.]

<u>Work by radial pressure p.</u>

This work equals p times decrease in volume.

If $w = 0$ at the ends, effects of end displacements can be ignored as arising from third-order changes of volume, since FG and F'G' are themselves of 2d order in u, w. (Fig. 3).

Any radius is shortened by $[w]_c$ or

$$w - u\, w_x - v\, w_s$$

Hence decrease in volume is, through quadratic terms,

$$\int \pi a^2\, dx - \int dx \int_0^{2\pi a} \tfrac{1}{2}(a - w + uw_x + vw_s)[(a - w + u\, w_x + vw_s)/a]ds$$

$$= \int dx \int (w - w^2/2a - u\, w_x - v\, w_s)\ ds$$

and work is

$$W_2 = p\int\int (w - w^2/2a - u\, w_x - v\, w_s)dxds\ . \qquad (\gamma)$$

[PIBAL 167 has in (25)

$$W_2 = p\int\int(w - w^2/2a + u_x\, w - aw\, w_{ss}/2)dxds. \qquad (\gamma')$$

The deduction is not well understood.]

Overall comparison with PIBAL 167.

For an overall comparison, assume, as in the applications, that P is due to p on end closures, so that (nearly enough)

$$P = ap/2h \ . \qquad (\delta)$$

Write $(\Delta U_1)_b$ for the part of ΔU_1 in (α) not containing P or p, this part agreeing with PIBAL 167.

Then, from (α) (β) (γ),

$$\Delta U_1 - W_1 - W_2 = (\Delta U_1)_b + ap\iint\left(\frac{w^2}{2a^2} - \frac{w_x^2}{4} - \frac{w_s^2}{2} + \frac{uw_x}{a} + \frac{vw_s}{a}\right)dxds \qquad (\varepsilon)$$

[PIBAL 167 would get, from $\Delta U_1 = (\Delta U_1)_b$ and (β', γ'),

$$\Delta U_1 - W_1 - W_2 = (\Delta U_1)_b$$
$$+ ap\iint\left(-\frac{w}{a} + \frac{w^2}{2a^2} + \frac{w\,w_{ss}}{2} - u_x w/a - u_x^2 - v_s^2 - w_s^2\right)dxds. \qquad (\varepsilon')$$

The first-order (i) term appearing here is eliminated by the special assumption that $\int_0^{2\pi} w\,d\varphi = 0$.]

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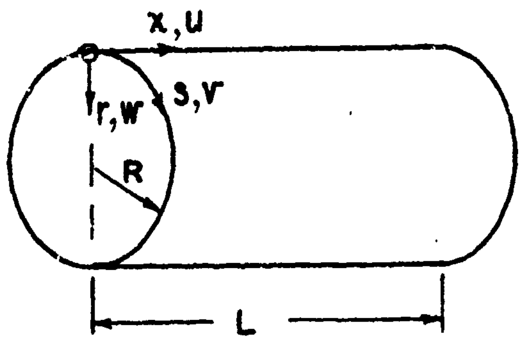

FIG. 1 COORDINATE SYSTEM AND DISPLACEMENTS FOR CIRCULAR CYLINDRICAL SHELL

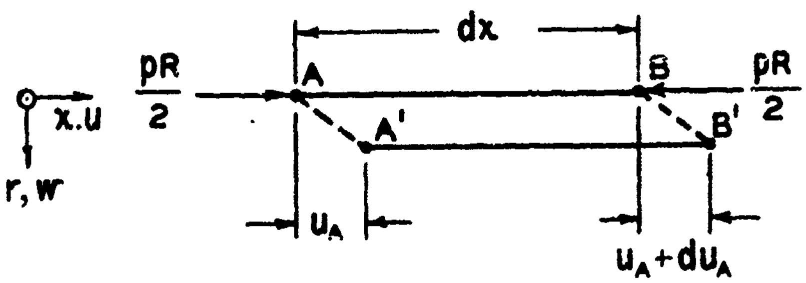

FIG. 2 AXIAL DISPLACEMENTS OF LONGITUDINAL ELEMENT OF SHELL

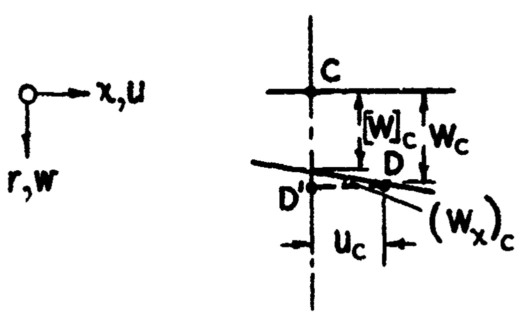

FIG. 3 DISPLACEMENT OF SHELL IN LONGITUDINAL PLANE